Differentiation using Product Rule and Quotient Rule

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Quick Basic Facts

- In Calculus,
- If $y = f(x)$ read as y is a function of x
- y is known as the dependent variable
- x is known as the independent variable

- Similarly if $y = f(s)$ read as y is a function of s
- y is known as the dependent variable
- s is known as the independent variable

All these point to…Differentiation

- Assume $y = f(x)$
- Find y'
- Find $f'(x)$
- Differentiate y wrt (with respect to) x
- Differentiate $f(x)$ wrt x
- Find the derivative of y wrt x
- Find the derivative of $f(x)$

Rules of Differentiation

- ***Power Rule***: if $y = ax^n$

  then $dy/dx = nax^{n-1}$

- ***Sum Rule***: if $y = u + v$

where y, u and v are functions of x

then $dy/dx = du/dx + dv/dx$

Rules of Differentiation

- ***Function of a Function Rule or Chain Rule***:
- If y is a function of u and u is a function of x,
- If y = f(u) and u = f(x),
- Then dy/dx = dy/du * du/dx
- We notice something here:
- y is not a direct function of x as we see in Power Rule and Sum Rule; rather
- y is a function of a variable, u; which in turn is a function of x

Rules of Differentiation

- ***Product Rule***: if y = u * v
- Where u and v are functions of x,
- then dy/dx = u* (dv/dx) + v * (du/dx)
- As a pneumonic, we can say that:
- Assume u = first and v = second, then
- dee y/dee x = [first * (dee second/dee x) +
  second * (dee first/dee x)]

So, in using the Product Rule,

- There must be a multiplication of terms or expressions which will result in a product (Product Rule)
- We have to always make sure that we have two terms (first term = u and second term = v)
- If we do not have two terms, we have to chunk the number of terms that we have, into two terms
- Depending on the question, it may be necessary to use other rules in conjunction with the Product Rule

Can we do an example?

- Example 1: Find the derivative of
- f(x) = sin x cos x using the product rule and simplify.
- Solution: f(x) = sin x cos x = sin x * cos x
- We have two terms: sin x = u = first term and

  cos x = v = second term

So, u = sin x; du/dx = cos x

v = cos x; dv/dx = -sin x

Applying the Product Rule, we have

- dy/dx = [(u * dv/dx) + (v * du/dx)]
- Thus, dy/dx = [(sin x * -sin x) + (cos x * cos x)]
- dy/dx = $-\sin^2 x + \cos^2 x$
- **dy/dx = $\cos^2 x - \sin^2 x$**
- This becomes our final answer.
- Please ask your questions.
- Ok, Let us move to another rule: the
- <u>Quotient Rule</u>

Rules of Differentiation

- ***Quotient Rule***: if y = u/v
- Where y, u and v are functions of x,
- Then dy/dx = [v * (du/dx) - u * (dv/dx)] / v^2
- As a pneumonic, we can say that:
- Assume u = top, and v = bottom; then
- dee y / dee x =
- [bottom * (dee top/dee x) – top * (dee bottom/dee x)]

  bottom squared

So in using the Quotient Rule,

- There must be a division of terms or expressions which will result in a quotient (Quotient Rule)
- Our numerator is always our top
- Our denominator is always the bottom
- Depending on the question, it may be necessary to use other rules in conjunction with the Quotient Rule

Can we do an example?

- Example 2: Find the derivative of
- $f(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$

using the quotient rule and simplify.

Solution: Top = u = $\sqrt{x} - 1$; u = $x^{1/2} - 1$

$du/dx = (1/2) * x^{1/2-1} - 0$[Power and Sum Rules]

$du/dx = (1/2)\, x^{-1/2}$

$du/dx = \dfrac{1}{2x^{1/2}}$

$du/dx = \dfrac{1}{2\sqrt{x}}$

We are getting there…

- Bottom = v = $\sqrt{x} + 1$; $v = x^{1/2} + 1$

$dv/dx = (1/2) * x^{1/2-1} + 0$

$dv/dx = (1/2)\, x^{-1/2}$

$$dv/dx = \frac{1}{2x^{1/2}}$$

$$dv/dx = \frac{1}{2\sqrt{x}}$$

Bottom squared = $(\sqrt{x} + 1)^2$

Applying the Quotient Rule, we now have

$$dy/dx = \frac{(\sqrt{x} + 1) * \frac{1}{2\sqrt{x}} - (\sqrt{x} - 1) * \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

$$dy/dx = \frac{\frac{\sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

Step by step, we will get our answer

$$\frac{dy}{dx} = \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2} - \frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{2}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

We are almost there…

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)\,(\sqrt{x} + 1)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}(x + 2\sqrt{x} + 1)} = \frac{1}{x\sqrt{x} + 2x + \sqrt{x}}$$

Simplifying gives

$$dy/dx = \frac{1}{\sqrt{x(x+1)} + 2x}$$

$$dy/dx = \frac{1}{\sqrt{x(x+1)} + 2x} * \frac{\sqrt{x(x+1)} - 2x}{\sqrt{x(x+1)} - 2x}$$

$$dy/dx = \frac{\sqrt{x(x+1)} - 2x}{[\sqrt{x(x+1)}]^2 - (2x)^2}$$

$$dy/dx = \frac{\sqrt{x(x+1)} - 2x}{\sqrt{x(x+1)} * \sqrt{x(x+1)} - 4x^2}$$

Simplifying further gives

$$\frac{dy}{dx} = \frac{\sqrt{x}(x + 1) - 2x}{\sqrt{x} * \sqrt{x} *(x + 1) * (x + 1) - 4x^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x}(x + 1) - 2x}{x(x + 1)^2 - 4x^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x}(x + 1) - 2x}{x(x + 1)(x + 1) - 4x^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x}(x + 1) - 2x}{x^3 + 2x^2 + x - 4x^2}$$

And our final answer is

$$dy/dx = \frac{\sqrt{x}(x + 1) - 2x}{x^3 + 2x^2 - 4x^2 + x}$$

$$dy/dx = \frac{\sqrt{x}(x + 1) - 2x}{x^3 - 2x^2 + x}$$

$$\mathbf{dy/dx = \frac{\sqrt{x}(x + 1) - 2x}{x(x^2 - 2x + 1)}}$$

Thank you for the opportunity

- Please ask your questions.